Klausur

MSc Volkswirtschaftslehre
"Dynamische Modelle in den Wirtschaftswissenschaften"
Wintersemester 2020/21, 1.2.2021

Lösen Sie drei (und nicht mehr als drei) der folgenden Aufgaben 1 – 5 und außerdem die beiden Aufgaben 6 und 7.

Die Gewichte bei der Bewertung sind 10 % für jede der Aufgaben 1 – 5, 25% für die Aufgabe 6 und 45 % für die Aufgabe 7.

"open book": Schriftliche Unterlagen sind als Hilfsmittel erlaubt.

Bei keiner der Aufgaben sind längere Texte erforderlich.

1. Lösen sie folgende **Differenzengleichung** und erläutern Sie den Rechenweg in kurzen Sätzen:

   $x_t - x_{t-1} = t - 1$, $x_0 = 0$

2. Eine **Differentialgleichung erster Ordnung** lautet $\dot{x} = -x^2 e^t$ mit $x(0) = 1$. Lösen Sie die Gleichung!

3. Betrachten Sie die Differentialgleichung $\dot{x} = 1 + x/a$ und zeichnen Sie das Phasen diagramm für a=0, a>0 und a<0. Zeichnen Sie dann das **Bifurkationsdiagramm**.

   Hinweis: Es ist hilfreich, zunächst den Gleichgewichtswert von x zu bestimmen-

4. Zeichnen Sie ein **Phasendiagramm** für das Gleichungssystem $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x} - \bar{\mathbf{x}})$ für den Fall.

   $$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

   Welche Eigenwerte erhält man? Um welchen Typ von Lösung (stabil, instabil, Fokus, Knoten, Sattelpfad …) handelt es sich? Weitere Erläuterungen sind nicht erforderlich.

5. Zeichnen Sie ein **Phasendiagramm** für das Gleichungssystem $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x} - \bar{\mathbf{x}})$ für den Fall

   $$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

   Welche Eigenwerte erhält man? Um welchen Typ von Lösung (stabil, instabil, Fokus, Knoten, Sattelpfad …) handelt es sich? Weitere Erläuterungen sind nicht erforderlich.

6 Nichtlineares Differentialgleichungssystem. In einem Ökosystem gibt es Hasen und Füchse. Die Hasen, $H(t)$, vermehren sich gemäß einer buckelförmigen konkaven Regenerationsfunktion $g(H(t))$ und werden durch die Füchse, $F(t)$ dezimiert, wobei jeder Fuchs den Hasenbestand um genau eine Einheit verringert. Schreiben Sie die entsprechende Differentialgleichung auf. Die Fuchspopulation entwickelt sich gemäß der Gleichung $\dot{F}(t) = a(F(t)\big(bH(t) - F(t)\big)$, wobei a und b positive Parameter sind.

Bestimmen Sie das Gleichgewicht und zeichnen Sie ein Phasendiagramm, so dass es ein ökologisches Gleichgewicht mit positiven Populationen gibt. Welche Stabilitätseigenschaften erkennen Sie beim näheren Hinschauen? Warum können solche optischen Eindrücke gerade im hier vorliegenden Fall irreführend sein, und wie würde man vorgehen, um exaktere Informationen zu gewinnen? Sie sollen diese <u>Berechnungen nicht durchführen</u>, sondern nur sagen, was man tun würde.

7 Dynamische Optimierung. Der Besucher einer Party möchte dort alkoholische Getränke konsumieren, $C(t)$, und zieht daraus einen Nutzen, $U\big(C(t)\big)$ mit $U' > 0, U'' < 0$, während der Zeit 0 bis T, die die Party dauert. Am nächsten Morgen sind Kopfschmerzen zu erwarten, die davon abhängen, wie alkoholisiert der Besucher im Zeitpunkt T ist. Dies wird abgebildet durch eine streng konvexe Kostenfunktion $K\big(A(T)\big)$ und geht negativ in die Zielfunktion ein. Alkohol wird im Körper mit der Rate δ abgebaut, so dass gilt $\dot{A}(t) = C(t) - \delta A(t)$. $A(0) = 0$. Da es nur um wenige Stunden geht, wird nicht diskontiert.

Schreiben Sie das Optimierungsproblem auf. Erstellen Sie dann die Hamiltonfunktion, leiten Sie die Optimalitätsbedingungen her, und eliminieren Sie die Kozustandsvariable. Warum ist die Kozustandsvariable negativ? Wie ist die Endbedingung für t=T zu interpretieren? Warum sinkt der Konsum während der Party?

Zeichnungen sind nicht zu erstellen.

1 Solution of an *n*-th order linear difference equation

1. Write the equation in the following form:

$$x_t + a_1 x_{t-1} + \ldots.. + a_n x_{t-n} = g(t).$$

2. Find the **particular solution** to the non-homogeneous equation.

   2.1 Assume that the solution is $\bar{x}_t = h(t)$, where $h(t)$ "resembles" $g(t)$.

   E.g., if $g(t)=gt$, use $\bar{x}_t = ht$.

   2.2 Solve the resulting equation for the parameter(s) of $h(t)$.

   2.3 If the solution involves a division by zero, use $\bar{x}_t = th(t)$. If this dos not help, use $\bar{x}_t = t^2 h(t)$, and so on ....

   2.4 The particular solution can be interpreted as a long-run equilibrium.

3. Find the **solution of the homogeneous equation**

   3.1 Assume that the solution is $\bar{x}_t = \lambda^t$. Thus:

$$\lambda^t + a_1\lambda^{t-1} + \ldots.. + a_n \lambda^{t-n} = 0,$$

   3.2 Rewrite this as

$$\lambda^{t-n}(\lambda^n + a_1\lambda^{n-1} + \ \ldots + a_{n-1}\lambda + a_n) = 0$$

   3.3 Solve the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + \ \ldots + a_{n-1}\lambda + a_n = \ 0.$$

   3.4 The solution of the difference equation is a linear combination of all possible solutions, λ_i

$$x_t = A_1\lambda_1^t + \ \ldots. + A_n\lambda_n^t.$$

   In the case of a quadratic characteristic equation, a negative discriminate may lead to conjugate complex solutions. Then

$$x_t = a_2^{t/2}(A_1 \cos\omega\, t + A_2 \sin\omega\, t).$$

   with

$$\cos\omega = \frac{a_1}{2\sqrt{a_2}} \quad , \quad \sin\omega = \sqrt{1 - \frac{a_1^2}{4a_2}} \quad .$$

4. **Stability**: The λ_i with the largest absolute value determines the stability behavior of the equation in the long run: stable if $|\lambda_{\mathrm{i}}^{max}\ | < 1$, unstable if $|\lambda_{\mathrm{i}}^{max}\ | > 1$.

5. To find the **general solution to the non-homogeneous equation**, write

$$x_t = A_1\lambda_1^t + \ \ldots. + A_n\lambda_n^t + \bar{x}_t.$$

   and use the information on particular values of x_t at n different points in time to determine the A_is.

2 Solution of a system of linear difference equations

The system is given as a set of n equations, the i-th equation being

$$x_{it} = a_{i1}\, x_{1t-1} + \ldots.. + a_{in}\, x_{nt-1} + g_i(t).$$

1. Write the system in matrix form

$$\mathbf{x_t} = \mathbf{A}\mathbf{x_{t-1}} + \mathbf{g}(t)$$

with

$$\mathbf{x_t} = \begin{pmatrix} x_{1t} \\ \vdots \\ x_{nt} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \qquad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$

2. Note that **finding a particular solution** is often impossible if the g_i functions are of different types. Find the particular solution $\bar{\mathbf{x}}_\mathbf{t}$ by assuming that $\bar{\mathbf{x}}_\mathbf{t}$has the same shape as $\mathbf{g}(t)$**.** If for example there is a vector of exponential functions, $\mathbf{g}(t) = (1+\delta)^t\mathbf{g}$, where g is a vector of constants, then try $\bar{\mathbf{x}}_\mathbf{t} = (1+\delta)^t\boldsymbol{\gamma}$ as a particular solution, where the $\boldsymbol{\gamma}$ is a vector of unknown constants. Using this in the difference equation, we have

$$\boldsymbol{\gamma} = \left(\mathbf{I} - \frac{\mathbf{A}}{1+\delta}\right)^{-1}\mathbf{g}.$$

Alternatively one may prefer to write a system of n equations and solve it for the γ_is by Cramer's rule. If $|(1+\delta)\mathbf{I} - \mathbf{A}| = 0$, the system does not have a solution and we try $\bar{\mathbf{x}}_{\boldsymbol{t}} = t\bar{\mathbf{x}}$. With other functions $\mathbf{g}(t)$, proceed in an analogous fashion.

3. Find the **solution of the homogeneous equations** by determining the eigenvalues of **A**. They can be found by solving det(**A** – λ**I**)=0. If n=2, they are

$$\lambda_{1,2} = \frac{\mathrm{tr}(\mathbf{A})}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A})}.$$

In the general case of n equations, the solution of the homogeneous equations is

$$x_{it} = A_1\beta_i^{(1)}\lambda_1^t + \ldots + A_n\beta_i^{(n)}\lambda_n^t,$$

where $i = 1, \ldots, n$ and where $\beta_i^{(j)}$ denotes the i-th component of the eigenvector corresponding to the j-th eigenvalue. The first components of the eigenvectors are set equal to one: $\beta_1^{(j)} = 1\ \forall j = 1, \ldots, n$. In the two-equations case we have

$$x_{1t} = A_1\lambda_1^t + A_2\lambda_2^t,$$

$$x_{2t} = A_1\frac{\lambda_1 - a_{11}}{a_{12}}\lambda_1^t + A_2\frac{\lambda_2 - a_{11}}{a_{12}}\lambda_2^t.$$

4. Write down the **solution of the homogeneous equations** adding $\bar{\mathbf{x}}_{\boldsymbol{t}}$ on the right-hand side.

5. **Determine the constants** by inserting the known values of $\mathbf{x_t}$ on the left-hand side and the corresponding values of t on the right-hand side.

6. The system is **stable** if $|\lambda_i| < 1$ for all $i \in (0, n)$. In the case of complex eigenvalues, $|\lambda_i| = \sqrt{\alpha^2 + \theta^2}$, where α is the real part and θ is the imaginary part.

3 Nonlinear difference equations and chaos in economic models

1. To construct a **phase diagram for a nonlinear difference equation**, write x_t on the vertical axis and x_{t-1} on the horizontal axis. Then draw $x_t = f(x_{t-1})$ and a 45° line into the diagram. Use $f(x_{t-1})$ to determine x_t and use the 45° line to obtain the starting value for the next period from x_t.

2. **Stability** can be checked by determining the slope of $f(x_{t-1})$ in the equilibrium. If the absolute value of the slope is larger than 1, the equilibrium is unstable.

3. **To construct a model that possibly generates chaos**, specify the functions of the underlying economic model such that you get a hump-shaped transition function $x_t = f(x_{t-1})$.

4 Differential equations

4.1 Linear differential equations with constant coefficients

Consider the differential equation:

$$\dot{x} + ax = g(t).$$ [1]

1. To **find a particular solution** of the non-homogeneous equation, assume that $\bar{x}(t)$ has the same functional form as $g(t)$, e.g. constant, linear, or exponential. Then insert this into the differential equation such that $\dot{\bar{x}} + a\bar{x} = g(t)$. Solve it. Usually this can be done without integration. If the solution involves division by zero, multiply the solution candidate by *t* and try again.

2. To **solve the homogeneous part**, rewrite it as:

$$\frac{dx}{dt} = -ax.$$

   Multiply by dt and divide by x:

$$\frac{dx}{x} = -adt.$$

   Integrate and obtain an equation which is logarithmic in *x* and linear in *t*. Solve this for *x*

$$x = A \cdot e^{-at},$$

   where $A = e^{A_t - A_x}$. This is the solution of the homogeneous equation.

3. The solution is **stable** if $a > 0$.

4. Use a given $x(t^*)$ to **eliminate the integration constant**.

[1] Sometimes a nonlinear differential equation can be transformed into a linear one, e.g in the case of Solows growth model, $\dot{k} = sk^{\alpha} - nk$. Replace *k* by $z^{1/(1-\alpha)}$. Then $\dot{z} = (1-\alpha)s - (1-\alpha)nz$.

4.2 Method of variation of the constant

If the particular solution of the non-homogeneous equation cannot be found by the standard method, try this approach.

1. Starting from the solution of the homogeneous equation, $x = A \cdot e^{-at}$, **replace the constant *A* by a function of time, *A*(*t*).** Use this in the original differential equation:

$$\dot{A} \cdot e^{-at} - aA \cdot e^{-at} + aA \cdot e^{-at} = g(t).$$

   The last two terms on the left-hand side cancel out and we have

$$\dot{A} = e^{at} g(t).$$

2. **Integration** then yields

$$A = \int g(t) e^{at}\, dt + \tilde{A},$$

   where $\tilde{A}$ is the new integration constant. We then have

$$x = \left(\int g(t) e^{at}\, dt + \tilde{A} \right) e^{-at}.$$

3. **Solve this integral** (if possible) and **eliminate the integration constant** by using a given $x(t^*)$.

4.3 Separable differential equations

Consider a differential equation of the type

$$f(x) \cdot \dot{x} = g(t).$$

1. **Rewrite the equation** by multiplying by dt and then integrating

$$\int f(x) dx + A_x = \int g(t) dt + A_t.$$

2. **Solve the integral**, subtract one of the constants on both sides to obtain a single constant and **eliminate the constant** by using a given $x(t^*)$.

4.4 Linear differential equations with time-varying coefficients

The differential equation is

$$\dot{x} + h(t)x = g(t).$$

1. Let us initially look at the **homogeneous equation** and rewrite it

$$\frac{dx}{x} = -h(t)dt.$$

   Integration yields

$$\ln(x) + A_x = -\int h(t)\,dt + A_t$$

$$x = Ae^{-\int h(t)dt}.$$

2. The **solution to the non-homogeneous equation** can be found by the **method of variation of the constant**. Rewrite the solution of the homogeneous equation by setting $A = A(t)$ and $e^{-\int h(t)dt} = z(t)$.

$$x = A(t) \cdot z(t)\,.$$

   Use this in the non-homogeneous equation:

$$\dot{A}z + A\dot{z} + h(t)Az = g(t)$$

$$\dot{A}z + A \cdot (\dot{z} + h(t)z) = g(t).$$

   The term in brackets vanishes since z is proportional to the solution of the homogeneous equation. Thus we have

$$\dot{A} = \frac{g(t)}{z(t)}.$$

   Integration yields

$$A = \int \frac{g(t)}{z(t)}dt + \tilde{A} = \int \frac{g(t)}{e^{-\int h(t)dt}}dt + \tilde{A},$$

   where $\tilde{A}$ is an integration constant. Thus we get as the general solution of the non-homogeneous equation

$$x = e^{-\int h(t)dt} \cdot \left(\tilde{A} + \int \frac{g(t)}{e^{-\int h(t)dt}}dt\right).$$

3. The **integration constant** can be determined by inserting the known value of x on the left-hand side and the corresponding time on the right-hand side.

4.5 Phase diagrams for single differential equations and bifurcations

Let the original differential equation be $\dot{z} = g(z, t)$.

1. Find a transformation $x = \varphi(z, t)$, such that the **equation becomes autonomous**: $\dot{x} = f(x)$. In economics, this is often done by multiplying *x* by an exponential function of time.[2]

2. **Draw an ($\boldsymbol{x}$, $\dot{\boldsymbol{x}}$) diagram** with *x* on the horizontal axis and $\dot{x}$ on the vertical axis and draw the $f(x)$ function into the diagram. If the function cuts the horizontal axis from above, the equilibrium is stable. If it cuts the horizontal axis from below, the equilibrium is unstable.

3. To **find bifurcations**, change a parameter of the function $f(x)$ such that the function is shifted. If new equilibria emerge or if old ones vanish, there are bifurcations.

4. To **draw a bifurcation diagram**, write the shift parameter on the horizontal axis and the equilibrium value(s) of *x* on the vertical axis. Use solid lines to characterise stable equilibria and dotted lines to characterise unstable equilibria.

[2] The standard example in economics is Solow's growth model, where we divide by population and express all variable in per-capita terms.

5 Systems of differential equations

5.1 Systems of linear differential equations

A system of linear differential equations is written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$$

1. To find a **particular solution to the non-homogeneous equation**, assume that $\bar{\mathbf{x}}(t)$ has the same shape as $\mathbf{g}(t)$, use this in the differential equation, and solve for the unknown parameters of $\bar{\mathbf{x}}(t)$. If this involves a division by zero, multiply the solution candidate by t. A solution can usually be found only if the functions contained in $\mathbf{g}(t)$ are of the same type.

2. The **solution of the homogeneous equation** for the *i*-th variable is

$$x_i = A_1\beta_i^{(1)}e^{\lambda_1 t} + \ldots + A_n\beta_i^{(n)}e^{\lambda_n t},$$

   where λ_j is the *j*-th eigenvalue and $\beta_i^{(j)}$ is the *i*-th component of the corresponding eigenvector of the matrix **A**. Eigenvalues are determined by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

   which involves solving a polynomial of degree *n*. Eigenvectors are then found by solving

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\beta} = 0$$

   and setting the first element of each eigenvector equal to 1:

3. In the special case of a **system of two equations**, we have

$$\lambda_{1,2} = \frac{\mathrm{tr}\mathbf{A}}{2} \pm \sqrt{\frac{(\mathrm{tr}\mathbf{A})^2}{4} - \det\mathbf{A}}.$$

   The solution of the homogeneous equation is

$$x_1 = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}$$

$$x_2 = A_1\frac{\lambda_1 - a_{11}}{a_{12}}e^{\lambda_1 t} + A_2\frac{\lambda_2 - a_{11}}{a_{12}}e^{\lambda_2 t}.$$

   The system is stable if $\lambda_1 < 0$ and $\lambda_2 < 0$. If the eigenvalues are complex, i.e. $\lambda_{1,2} = \alpha \pm \theta i$, where $i = \sqrt{-1}$, then we have cyclical solutions:

$$x_1 = e^{\alpha t}(A_1\cos\theta t + A_2\sin\theta t)$$

$$x_2 = e^{\alpha t}\left(\frac{(\alpha - a_{11})A_1 + \theta A_2}{a_{12}}\cos\theta t + \frac{(\alpha - a_{11})A_2 + \theta A_1}{a_{12}}\sin\theta t\right).$$

   The cyclical paths are stable if $\alpha < 0$, i.e. if tr**A** < 0.

4. To find the **general solution of the non-homogeneous equations**, add the particular solution to the general solution of the homogeneous equation. For the *i*-th equation, we have

$$x_i = A_1\beta_i^{(1)}e^{\lambda_1 t} + \ldots + A_n\beta_i^{(n)}e^{\lambda_n t} + \bar{x}_i,$$

5. Finally **determine the integration constants** by inserting the known values of x_i on the left-hand side and the corresponding values of t on the right-hand side.

5.2 Linearisation of nonlinear systems of differential equations and stability analysis

A system of nonlinear differential equations can be linearised by using a first-order Taylor approximation. The i-th equation of the homogeneous system is

$$\dot{x}_i = f^i(x_1, \ldots, x_n, t)$$

1. Determine the **Jacobian matrix J** with the elements $a_{ij} = \frac{\partial f^i}{\partial x_j}$.
2. The **linerarised system** is $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x} - \bar{\mathbf{x}})$.
3. **Stability** is determined by the eigenvalues of **J**, where the results are only valid in a neighbourhood of the equilibrium. See Section 5.4 below for the stability conditions.

5.3 Rules for stability of systems of two equations

a) Real eigenvalues, $\lambda_1 < 0$ and $\lambda_2 < 0$. Stable node: non-cyclical paths.
b) Real eigenvalues, $\lambda_1 < 0$ and $\lambda_2 > 0$. Saddle point: monotonous saddle path.
c) Real eigenvalues, $\lambda_1 > 0$ and $\lambda_2 > 0$. Unstable node, non-cyclical paths.
d) Complex eigenvalues, negative real parts. Stable focus, cyclical paths.
e) Complex eigenvalues, zero real parts. Closed orbits.
f) Complex eigenvalues, positive real parts. Unstable node: cyclical paths.

5.5 Generalisation of the rules for general systems of equations

a) If there are complex eigenvalues, the solution is cyclical.
b) If all real eigenvalues are negative and all complex eigenvalues have negative real parts, the system is stable.
c) If k eigenvalues are negative or have negative real parts and $n - k$ eigenvalues are positive or have positive real parts, there is a k-dimensional stable manifold, leading to

the equilibrium. For the system to be on this manifold, only k starting values of $\mathbf{x}$ can be chosen arbitrarily. The remaining and $n - k$ ones have to take specific values.

5.6 Phase diagrams for systems of two equations.

The system to be considered is

$$\dot{x}_1 = f^1(x_1, x_2)$$
$$\dot{x}_2 = f^2(x_1, x_2)$$

1. Phase diagrams can be drawn only for autonomous systems like the one above. If the system is non-autonomous, it needs to be **transformed into an autonomous system** via a transformation of variables. In economic models, the non-autonomous part of a differential equation often consists of an exponential function of time. The original variables are then transformed by multiplying them by an exponential function of time. In models with population growth is this is done by moving to per-capita variables.

2. **Draw an $(\boldsymbol{x_1}, \boldsymbol{x_2})$ diagram** with x_1 on the horizontal axis and x_2 on the vertical axis. Of course, this can be reversed if appropriate.

3. **Draw the isoclines** $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. If $f^1(x_1, x_2)$ or $f^2(x_1, x_2)$ are implicit functions, use the implicit-function theorem to determine the shape of the isoclines.

4. **Equilibria** are determined by $f^1(x_1, x_2) = f^2(x_1, x_2) = 0$. In the case on nonlinear functions, there may be multiple equilibria.

5. **Draw arrows that visualise the movements** in horizontal and vertical direction. Draw horizontal-vertical combinations of these arrows into all sectors that are separated from each other by isoclines.

6. Visualise the **direction of motion of trajectories crossing the isoclines**. Where do they come from, where do they go? In the intersection points with the $\dot{x}_1 = 0$ isocline, all trajectories are vertical. In the intersection points with the $\dot{x}_2 = 0$ isocline, they are horizontal.

7. From this information one may infer the **stability pattern**: cyclical vs. non-cyclical, stable vs. unstable, and if unstable saddle path or complete instability. **Caveat:** foci and nodes are not always easily distinguished. A formal stability analysis using the eigenvalues of the Jacobian may be helpful.

6 Dynamic optimisation and Pontryagin's maximum principle

The problem is to maximise

$$\int_0^T e^{-\rho t} f(x,u)dt + e^{-\rho T} F\big(x(T)\big)$$

with respect to $u(t)$ and – possibly – with respect to T and subject to

$$\dot{x} = g(x,u),$$

$$x(0) = x.$$

x and u may be vectors. $f(x,u)$ and $g(x,u)$ are assumed to be concave.

1. Write the **current-value Hamiltonian**

   $$H = f(x,u) + \lambda g(x,u),$$

   If x is a vector of length n, there will be a vector λ of costate variables of length n as well.

2. Determine the **optimality conditions**, i.e. the first-order condition(s) with respect to u and the canonical equation(s):

   $$H_u = 0.$$

   $$\dot{\lambda} = \rho\lambda - H_x$$

   If u is a vector of length k, there will be k optimality conditions. If x is a vector of length n, there will be n canonical equations.

3. **Eliminate the costate variable(s).**

   3.1 **Differentiate the first-order condition** with respect to time. This gives an equation in which $\dot{\lambda}$ occurs.

   3.2 Use the canonical equation to **eliminate $\dot{\lambda}$**.

   3.3 Insert for λ from the first-order condition to **eliminate λ**.

   The result is a differential equation $\dot{u} = \varphi(x,u)$. This equation may be quite complex if the functions $f(x,u)$ and $g(x,u)$ contain interactions terms of x and u. Then there will be cross derivatives and they complicate the remainder of the analysis.

4. To **determine the equilibrium**, set $\dot{x} = 0$ and $\dot{u} = 0$. The $\dot{u} = 0$ condition often has the economic interpretation of a capital-market equilibrium condition or an arbitrage condition. Note that the equilibrium is not always unique.

5. To represent the solution, **draw a phase diagram** in the (x,u) space with x on the horizontal axis and u on the vertical axis. Proceed as follows

   - Start with the **isoclines**.

- Then draw the **rectangular combination of arrows** indicating horizontal and vertical motion.
- Then add parts of the **trajectories where they cross the isoclines**.
- Finally draw the **saddle path** (if there is one).

6. Several possibilities exist regarding the **time horizon** and the termination of the programme.

6.1 If the **time horizon is infinite** and if the Hamiltonian is strictly concave in (x,u), the optimal solution is the saddle path and the control variable "jumps" onto the saddle path at t=0.

6.2 If the **time horizon is finite** and if the **final state is evaluated** by $e^{-\rho T}F\big(x(T)\big)$, then $\lambda(T) = F'\big(x(T)\big)$. Inserting for $\lambda(T)$ from the first-order condition gives a relationship between $x(T)$ and $u(T)$ that can be represented as a line in the (x,u) space. The optimal path is the trajectory connecting the vertical $x=x_0$ line with this line within the given time T.

6.3 If the **time horizon is endogenous** and if the **final state is evaluated**, the terminal conditions are $H(T) - \rho F\big(x(T)\big) = 0$ and $\lambda(T) = F'\big(x(T)\big)$. Using these conditions, λ can be eliminated. This gives a point in the (x,u) space. The optimal path ends in this point. The optimal duration of the programme is determined by the time needed to move from the $x=x_0$ line to this termination point.

6.4 If the **time horizon is endogenous** and if the **final state is not evaluated**, the terminal conditions are $H(T) = 0$ and $e^{-\rho T}\lambda(T) = 0$. There are models in which the second condition holds only for $T \to \infty$.

Michael Rauscher

Cook Recipes for Dynamic Models in Economics

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6 Dynamic optimisation and Pontryagin's maximum principle

1 Solution of an *n*-th order linear difference equation

6. Write the equation in the following form:

$$x_t + a_1 x_{t-1} + \ldots.. + a_n x_{t-n} = g(t).$$

7. Find the **particular solution** to the non-homogeneous equation.

   7.1 Assume that the solution is $\bar{x}_t = h(t)$, where $h(t)$ "resembles" $g(t)$.

   E.g., if $g(t)=gt$, use $\bar{x}_t = ht$.

   7.2 Solve the resulting equation for the parameter(s) of $h(t)$.

   7.3 If the solution involves a division by zero, use $\bar{x}_t = th(t)$. If this dos not help, use $\bar{x}_t = t^2 h(t)$, and so on ....

   7.4 The particular solution can be interpreted as a long-run equilibrium.

8. Find the **solution of the homogeneous equation**

   8.1 Assume that the solution is $\bar{x}_t = \lambda^t$. Thus:

$$\lambda^t + a_1\lambda^{t-1} + \ldots.. + a_n \lambda^{t-n} = 0,$$

   3.2 Rewrite this as

$$\lambda^{t-n}(\lambda^n + a_1\lambda^{n-1} + \ \ldots + a_{n-1}\lambda + a_n) = 0$$

   3.3 Solve the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + \ \ldots + a_{n-1}\lambda + a_n = \ 0.$$

   3.4 The solution of the difference equation is a linear combination of all possible solutions, λ_i

$$x_t = A_1\lambda_1^t + \ \ldots. + A_n\lambda_n^t.$$

   In the case of a quadratic characteristic equation, a negative discriminate may lead to conjugate complex solutions. Then

$$x_t = a_2^{t/2}(A_1 \cos\omega\, t + A_2 \sin\omega\, t).$$

   with

$$\cos\omega = \frac{a_1}{2\sqrt{a_2}} \ \ , \ \ \sin\omega = \sqrt{1 - \frac{a_1^2}{4a_2}} \ \ .$$

9. **Stability**: The λ_i with the largest absolute value determines the stability behavior of the equation in the long run: stable if $|\lambda_{\mathrm{i}}^{max}\ | < 1$, unstable if $|\lambda_{\mathrm{i}}^{max}\ | > 1$.

10. To find the **general solution to the non-homogeneous equation**, write

$$x_t = A_1\lambda_1^t + \ \ldots. + A_n\lambda_n^t + \bar{x}_t.$$

   and use the information on particular values of x_t at n different points in time to determine the A_is.

5 Solution of a system of linear difference equations

The system is given as a set of n equations, the i-th equation being

$$x_{it} = a_{i1}\, x_{1t-1} + \ldots.. + a_{in}\, x_{nt-1} + g_i(t).$$

4. Write the system in matrix form

$$\mathbf{x_t} = \mathbf{A}\mathbf{x_{t-1}} + \mathbf{g}(t)$$

with

$$\mathbf{x_t} = \begin{pmatrix} x_{1t} \\ \vdots \\ x_{nt} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \qquad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$

5. Note that **finding a particular solution** is often impossible if the g_i functions are of different types. Find the particular solution $\bar{\mathbf{x}}_\mathbf{t}$ by assuming that $\bar{\mathbf{x}}_\mathbf{t}$has the same shape as $\mathbf{g}(t)$**.** If for example there is a vector of exponential functions, $\mathbf{g}(t) = (1+\delta)^t\mathbf{g}$, where g is a vector of constants, then try $\bar{\mathbf{x}}_\mathbf{t} = (1+\delta)^t\boldsymbol{\gamma}$ as a particular solution, where the $\boldsymbol{\gamma}$ is a vector of unknown constants. Using this in the difference equation, we have

$$\boldsymbol{\gamma} = \left(\mathbf{I} - \frac{\mathbf{A}}{1+\delta}\right)^{-1}\mathbf{g}.$$

Alternatively one may prefer to write a system of n equations and solve it for the γ_is by Cramer's rule. If $|(1+\delta)\mathbf{I} - \mathbf{A}| = 0$, the system does not have a solution and we try $\bar{\mathbf{x}}_{\boldsymbol{t}} = t\bar{\mathbf{x}}$. With other functions $\mathbf{g}(t)$, proceed in an analogous fashion.

6. Find the **solution of the homogeneous equations** by determining the eigenvalues of **A**. They can be found by solving det(**A** – λ**I**)=0. If n=2, they are

$$\lambda_{1,2} = \frac{\mathrm{tr}(\mathbf{A})}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(\mathbf{A})}{2}\right)^2 - \det(\mathbf{A})}.$$

In the general case of n equations, the solution of the homogeneous equations is

$$x_{it} = A_1\beta_i^{(1)}\lambda_1^t + \ldots + A_n\beta_i^{(n)}\lambda_n^t,$$

where $i = 1, \ldots, n$ and where $\beta_i^{(j)}$ denotes the i-th component of the eigenvector corresponding to the j-th eigenvalue. The first components of the eigenvectors are set equal to one: $\beta_1^{(j)} = 1\ \forall j = 1, \ldots, n$. In the two-equations case we have

$$x_{1t} = A_1\lambda_1^t + A_2\lambda_2^t,$$

$$x_{2t} = A_1\frac{\lambda_1 - a_{11}}{a_{12}}\lambda_1^t + A_2\frac{\lambda_2 - a_{11}}{a_{12}}\lambda_2^t.$$

7. Write down the **solution of the homogeneous equations** adding $\bar{\mathbf{x}}_{\boldsymbol{t}}$ on the right-hand side.

8. **Determine the constants** by inserting the known values of $\mathbf{x_t}$ on the left-hand side and the corresponding values of t on the right-hand side.

9. The system is **stable** if $|\lambda_i| < 1$ for all $i \in (0, n)$. In the case of complex eigenvalues, $|\lambda_i| = \sqrt{\alpha^2 + \theta^2}$, where α is the real part and θ is the imaginary part.

6 Nonlinear difference equations and chaos in economic models

4. To construct a **phase diagram for a nonlinear difference equation**, write x_t on the vertical axis and x_{t-1} on the horizontal axis. Then draw $x_t = f(x_{t-1})$ and a 45° line into the diagram. Use $f(x_{t-1})$ to determine x_t and use the 45° line to obtain the starting value for the next period from x_t.

5. **Stability** can be checked by determining the slope of $f(x_{t-1})$ in the equilibrium. If the absolute value of the slope is larger than 1, the equilibrium is unstable.

6. **To construct a model that possibly generates chaos**, specify the functions of the underlying economic model such that you get a hump-shaped transition function $x_t = f(x_{t-1})$.

7 Differential equations

4.1 Linear differential equations with constant coefficients

Consider the differential equation:

$$\dot{x} + ax = g(t).$$ [3]

5. To **find a particular solution** of the non-homogeneous equation, assume that $\bar{x}(t)$ has the same functional form as $g(t)$, e.g. constant, linear, or exponential. Then insert this into the differential equation such that $\dot{\bar{x}} + a\bar{x} = g(t)$. Solve it. Usually this can be done without integration. If the solution involves division by zero, multiply the solution candidate by *t* and try again.

6. To **solve the homogeneous part**, rewrite it as:

$$\frac{dx}{dt} = -ax.$$

   Multiply by dt and divide by x:

$$\frac{dx}{x} = -adt.$$

   Integrate and obtain an equation which is logarithmic in *x* and linear in *t*. Solve this for *x*

$$x = A \cdot e^{-at},$$

   where $A = e^{A_t - A_x}$. This is the solution of the homogeneous equation.

7. The solution is **stable** if $a > 0$.

8. Use a given $x(t^*)$ to **eliminate the integration constant**.

[3] Sometimes a nonlinear differential equation can be transformed into a linear one, e.g in the case of Solows growth model, $\dot{k} = sk^{\alpha} - nk$. Replace *k* by $z^{1/(1-\alpha)}$. Then $\dot{z} = (1-\alpha)s - (1-\alpha)nz$.

8.2 Method of variation of the constant

If the particular solution of the non-homogeneous equation cannot be found by the standard method, try this approach.

4. Starting from the solution of the homogeneous equation, $x = A \cdot e^{-at}$, **replace the constant *A* by a function of time, *A*(*t*).** Use this in the original differential equation:

$$\dot{A} \cdot e^{-at} - aA \cdot e^{-at} + aA \cdot e^{-at} = g(t).$$

   The last two terms on the left-hand side cancel out and we have

$$\dot{A} = e^{at} g(t).$$

5. **Integration** then yields

$$A = \int g(t) e^{at}\, dt + \tilde{A},$$

   where $\tilde{A}$ is the new integration constant. We then have

$$x = \left(\int g(t) e^{at}\, dt + \tilde{A} \right) e^{-at}.$$

6. **Solve this integral** (if possible) and **eliminate the integration constant** by using a given $x(t^*)$.

4.3 Separable differential equations

Consider a differential equation of the type

$$f(x) \cdot \dot{x} = g(t).$$

3. **Rewrite the equation** by multiplying by dt and then integrating

$$\int f(x) dx + A_x = \int g(t) dt + A_t.$$

4. **Solve the integral**, subtract one of the constants on both sides to obtain a single constant and **eliminate the constant** by using a given $x(t^*)$.

4.4 Linear differential equations with time-varying coefficients

The differential equation is

$$\dot{x} + h(t)x = g(t).$$

4. Let us initially look at the **homogeneous equation** and rewrite it

$$\frac{dx}{x} = -h(t)dt.$$

Integration yields

$$\ln(x) + A_x = -\int h(t)\,dt + A_t$$

$$x = Ae^{-\int h(t)dt}.$$

5. The **solution to the non-homogeneous equation** can be found by the **method of variation of the constant**. Rewrite the solution of the homogeneous equation by setting $A = A(t)$ and $e^{-\int h(t)dt} = z(t)$.

$$x = A(t) \cdot z(t)\,.$$

Use this in the non-homogeneous equation:

$$\dot{A}z + A\dot{z} + h(t)Az = g(t)$$

$$\dot{A}z + A \cdot (\dot{z} + h(t)z) = g(t).$$

The term in brackets vanishes since z is proportional to the solution of the homogeneous equation. Thus we have

$$\dot{A} = \frac{g(t)}{z(t)}.$$

Integration yields

$$A = \int \frac{g(t)}{z(t)}dt + \tilde{A} = \int \frac{g(t)}{e^{-\int h(t)dt}}dt + \tilde{A},$$

where $\tilde{A}$ is an integration constant. Thus we get as the general solution of the non-homogeneous equation

$$x = e^{-\int h(t)dt} \cdot \left(\tilde{A} + \int \frac{g(t)}{e^{-\int h(t)dt}}dt\right).$$

6. The **integration constant** can be determined by inserting the known value of x on the left-hand side and the corresponding time on the right-hand side.

7.5 Phase diagrams for single differential equations and bifurcations

Let the original differential equation be $\dot{z} = g(z, t)$.

5. Find a transformation $x = \varphi(z, t)$, such that the **equation becomes autonomous**: $\dot{x} = f(x)$. In economics, this is often done by multiplying *x* by an exponential function of time.[4]

6. **Draw an ($\boldsymbol{x}$, $\dot{\boldsymbol{x}}$) diagram** with *x* on the horizontal axis and $\dot{x}$ on the vertical axis and draw the $f(x)$ function into the diagram. If the function cuts the horizontal axis from above, the equilibrium is stable. If it cuts the horizontal axis from below, the equilibrium is unstable.

7. To **find bifurcations**, change a parameter of the function $f(x)$ such that the function is shifted. If new equilibria emerge or if old ones vanish, there are bifurcations.

8. To **draw a bifurcation diagram**, write the shift parameter on the horizontal axis and the equilibrium value(s) of *x* on the vertical axis. Use solid lines to characterise stable equilibria and dotted lines to characterise unstable equilibria.

[4] The standard example in economics is Solow's growth model, where we divide by population and express all variable in per-capita terms.

5 Systems of differential equations

5.1 Systems of linear differential equations

A system of linear differential equations is written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$$

6. To find a **particular solution to the non-homogeneous equation**, assume that $\bar{\mathbf{x}}(t)$ has the same shape as $\mathbf{g}(t)$, use this in the differential equation, and solve for the unknown parameters of $\bar{\mathbf{x}}(t)$. If this involves a division by zero, multiply the solution candidate by t. A solution can usually be found only if the functions contained in $\mathbf{g}(t)$ are of the same type.

7. The **solution of the homogeneous equation** for the *i*-th variable is

$$x_i = A_1\beta_i^{(1)}e^{\lambda_1 t} + \ldots + A_n\beta_i^{(n)}e^{\lambda_n t},$$

where λ_j is the *j*-th eigenvalue and $\beta_i^{(j)}$ is the *i*-th component of the corresponding eigenvector of the matrix **A**. Eigenvalues are determined by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

which involves solving a polynomial of degree *n*. Eigenvectors are then found by solving

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\beta} = 0$$

and setting the first element of each eigenvector equal to 1:

8. In the special case of a **system of two equations**, we have

$$\lambda_{1,2} = \frac{\mathrm{tr}\mathbf{A}}{2} \pm \sqrt{\frac{(\mathrm{tr}\mathbf{A})^2}{4} - \det\mathbf{A}}.$$

The solution of the homogeneous equation is

$$x_1 = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}$$

$$x_2 = A_1\frac{\lambda_1 - a_{11}}{a_{12}}e^{\lambda_1 t} + A_2\frac{\lambda_2 - a_{11}}{a_{12}}e^{\lambda_2 t}.$$

The system is stable if $\lambda_1 < 0$ and $\lambda_2 < 0$. If the eigenvalues are complex, i.e. $\lambda_{1,2} = \alpha \pm \theta i$, where $i = \sqrt{-1}$, then we have cyclical solutions:

$$x_1 = e^{\alpha t}(A_1\cos\theta t + A_2\sin\theta t)$$

$$x_2 = e^{\alpha t}\left(\frac{(\alpha - a_{11})A_1 + \theta A_2}{a_{12}}\cos\theta t + \frac{(\alpha - a_{11})A_2 + \theta A_1}{a_{12}}\sin\theta t\right).$$

The cyclical paths are stable if $\alpha < 0$, i.e. if tr**A** < 0.

9. To find the **general solution of the non-homogeneous equations**, add the particular solution to the general solution of the homogeneous equation. For the *i*-th equation, we have

$$x_i = A_1\beta_i^{(1)}e^{\lambda_1 t} + \ldots + A_n\beta_i^{(n)}e^{\lambda_n t} + \bar{x}_i,$$

10. Finally **determine the integration constants** by inserting the known values of x_i on the left-hand side and the corresponding values of t on the right-hand side.

5.2 Linearisation of nonlinear systems of differential equations and stability analysis

A system of nonlinear differential equations can be linearised by using a first-order Taylor approximation. The i-th equation of the homogeneous system is

$$\dot{x}_i = f^i(x_1, \ldots, x_n, t)$$

1. Determine the **Jacobian matrix J** with the elements $a_{ij} = \frac{\partial f^i}{\partial x_j}$.
2. The **linerarised system** is $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x} - \bar{\mathbf{x}})$.
3. **Stability** is determined by the eigenvalues of **J**, where the results are only valid in a neighbourhood of the equilibrium. See Section 5.4 below for the stability conditions.

5.3 Rules for stability of systems of two equations

g) Real eigenvalues, $\lambda_1 < 0$ and $\lambda_2 < 0$. Stable node: non-cyclical paths.

h) Real eigenvalues, $\lambda_1 < 0$ and $\lambda_2 > 0$. Saddle point: monotonous saddle path.

i) Real eigenvalues, $\lambda_1 > 0$ and $\lambda_2 > 0$. Unstable node, non-cyclical paths.

j) Complex eigenvalues, negative real parts. Stable focus, cyclical paths.

k) Complex eigenvalues, zero real parts. Closed orbits.

l) Complex eigenvalues, positive real parts. Unstable node: cyclical paths.

5.5 Generalisation of the rules for general systems of equations

d) If there are complex eigenvalues, the solution is cyclical.

e) If all real eigenvalues are negative and all complex eigenvalues have negative real parts, the system is stable.

f) If k eigenvalues are negative or have negative real parts and $n - k$ eigenvalues are positive or have positive real parts, there is a k-dimensional stable manifold, leading to

the equilibrium. For the system to be on this manifold, only k starting values of $\mathbf{x}$ can be chosen arbitrarily. The remaining and $n - k$ ones have to take specific values.

5.6 Phase diagrams for systems of two equations.

The system to be considered is

$$\dot{x}_1 = f^1(x_1, x_2)$$
$$\dot{x}_2 = f^2(x_1, x_2)$$

1. Phase diagrams can be drawn only for autonomous systems like the one above. If the system is non-autonomous, it needs to be **transformed into an autonomous system** via a transformation of variables. In economic models, the non-autonomous part of a differential equation often consists of an exponential function of time. The original variables are then transformed by multiplying them by an exponential function of time. In models with population growth is this is done by moving to per-capita variables.

2. **Draw an $(\boldsymbol{x_1}, \boldsymbol{x_2})$ diagram** with x_1 on the horizontal axis and x_2 on the vertical axis. Of course, this can be reversed if appropriate.

3. **Draw the isoclines** $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. If $f^1(x_1, x_2)$ or $f^2(x_1, x_2)$ are implicit functions, use the implicit-function theorem to determine the shape of the isoclines.

4. **Equilibria** are determined by $f^1(x_1, x_2) = f^2(x_1, x_2) = 0$. In the case on nonlinear functions, there may be multiple equilibria.

5. **Draw arrows that visualise the movements** in horizontal and vertical direction. Draw horizontal-vertical combinations of these arrows into all sectors that are separated from each other by isoclines.

6. Visualise the **direction of motion of trajectories crossing the isoclines**. Where do they come from, where do they go? In the intersection points with the $\dot{x}_1 = 0$ isocline, all trajectories are vertical. In the intersection points with the $\dot{x}_2 = 0$ isocline, they are horizontal.

7. From this information one may infer the **stability pattern**: cyclical vs. non-cyclical, stable vs. unstable, and if unstable saddle path or complete instability. **Caveat:** foci and nodes are not always easily distinguished. A formal stability analysis using the eigenvalues of the Jacobian may be helpful.

7 Dynamic optimisation and Pontryagin's maximum principle

The problem is to maximise

$$\int_0^T e^{-\rho t} f(x,u)dt + e^{-\rho T} F\big(x(T)\big)$$

with respect to $u(t)$ and – possibly – with respect to T and subject to

$$\dot{x} = g(x,u),$$

$$x(0) = x.$$

x and u may be vectors. $f(x,u)$ and $g(x,u)$ are assumed to be concave.

7. Write the **current-value Hamiltonian**

$$H = f(x,u) + \lambda g(x,u),$$

If x is a vector of length n, there will be a vector λ of costate variables of length n as well.

8. Determine the **optimality conditions**, i.e. the first-order condition(s) with respect to u and the canonical equation(s):

$$H_u = 0.$$

$$\dot{\lambda} = \rho\lambda - H_x$$

If u is a vector of length k, there will be k optimality conditions. If x is a vector of length n, there will be n canonical equations.

9. **Eliminate the costate variable(s).**

   9.1 **Differentiate the first-order condition** with respect to time. This gives an equation in which $\dot{\lambda}$ occurs.

   9.2 Use the canonical equation to **eliminate $\dot{\lambda}$**.

   9.3 Insert for λ from the first-order condition to **eliminate λ**.

The result is a differential equation $\dot{u} = \varphi(x,u)$. This equation may be quite complex if the functions $f(x,u)$ and $g(x,u)$ contain interactions terms of x and u. Then there will be cross derivatives and they complicate the remainder of the analysis.

10. To **determine the equilibrium**, set $\dot{x} = 0$ and $\dot{u} = 0$. The $\dot{u} = 0$ condition often has the economic interpretation of a capital-market equilibrium condition or an arbitrage condition. Note that the equilibrium is not always unique.

11. To represent the solution, **draw a phase diagram** in the (x,u) space with x on the horizontal axis and u on the vertical axis. Proceed as follows

- Start with the **isoclines**.

- Then draw the **rectangular combination of arrows** indicating horizontal and vertical motion.
- Then add parts of the **trajectories where they cross the isoclines**.
- Finally draw the **saddle path** (if there is one).

12. Several possibilities exist regarding the **time horizon** and the termination of the programme.

12.1 If the **time horizon is infinite** and if the Hamiltonian is strictly concave in (x,u), the optimal solution is the saddle path and the control variable "jumps" onto the saddle path at t=0.

12.2 If the **time horizon is finite** and if the **final state is evaluated** by $e^{-\rho T}F\big(x(T)\big)$, then $\lambda(T) = F'\big(x(T)\big)$. Inserting for $\lambda(T)$ from the first-order condition gives a relationship between $x(T)$ and $u(T)$ that can be represented as a line in the (x,u) space. The optimal path is the trajectory connecting the vertical $x=x_0$ line with this line within the given time T.

12.3 If the **time horizon is endogenous** and if the **final state is evaluated**, the terminal conditions are $H(T) - \rho F\big(x(T)\big) = 0$ and $\lambda(T) = F'\big(x(T)\big)$. Using these conditions, λ can be eliminated. This gives a point in the (x,u) space. The optimal path ends in this point. The optimal duration of the programme is determined by the time needed to move from the $x=x_0$ line to this termination point.

12.4 If the **time horizon is endogenous** and if the **final state is not evaluated**, the terminal conditions are $H(T) = 0$ and $e^{-\rho T}\lambda(T) = 0$. There are models in which the second condition holds only for $T \to \infty$.