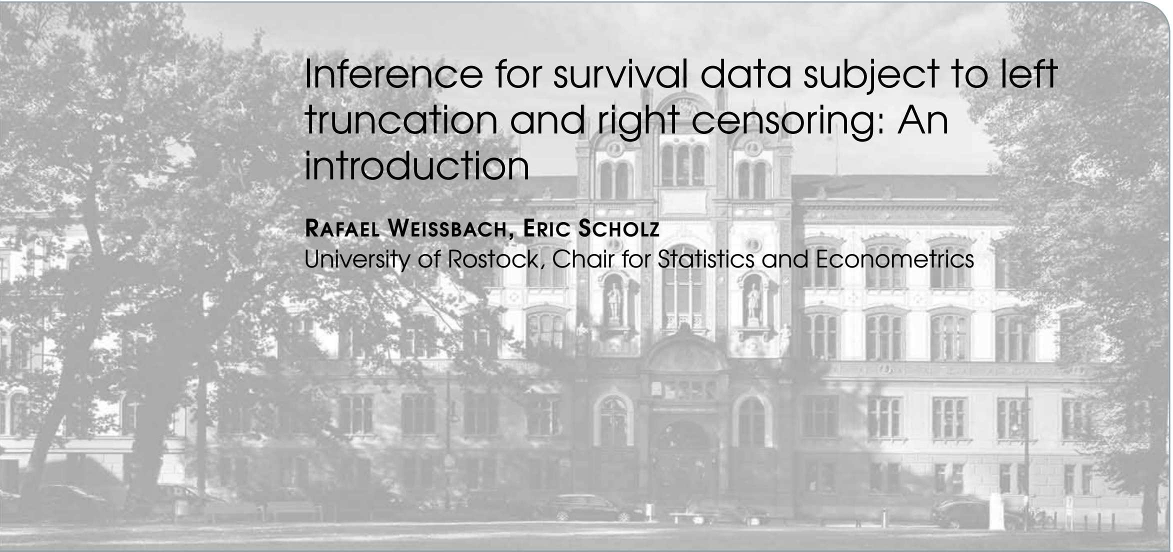
Universität Rostock Traditio et Innovatio



Inference for survival data subject to left truncation and right censoring: An introduction

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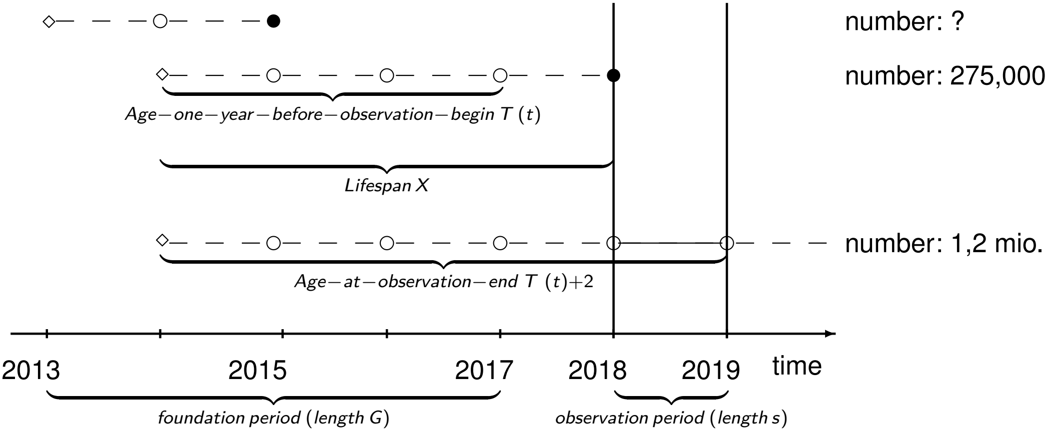
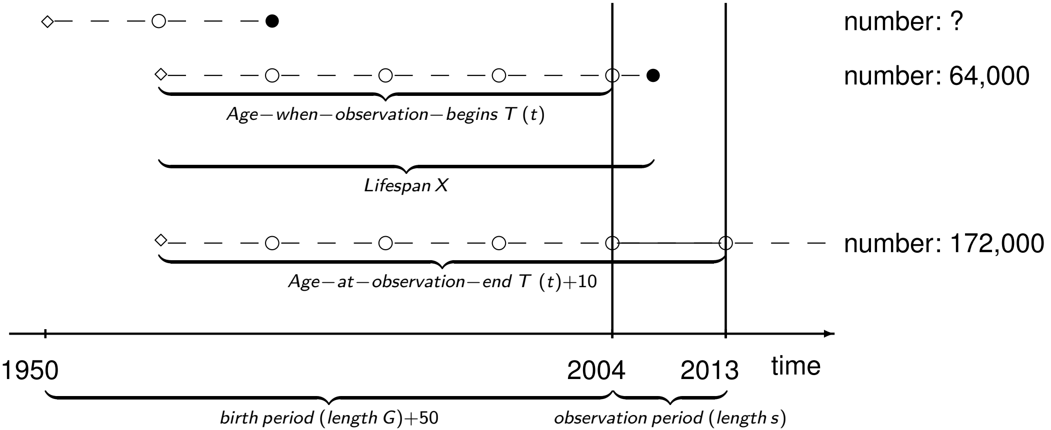
Foundation year	Year of closure 2018	2019	2020 or later
2013 ($t = 4$)	37	17	192
2014 ($t = 3$)	42	20	200
2015 ($t = 2$)	36	23	210
2016 ($t = 1$)	35	27	278
2017 ($t = 0$)	19	19	293
2013-2017	168	107	1,173
Total no. of observations (m): 1.4 mio			

Source: FDZ (Destatis)


number: ?
number: 275,000
Age – one – year – before – observation – begin T (t)
Lifespan X
number: 1,2 mio.
Age – at – observation – end T (t)+2
2013
2015
2017
2018
2019
time
foundation period (length G)
observation period (length s)

Year of birth†	Year of death 2004 ...	... 2013	2014 or later
1900-1953	64		172
Total no. of observations (m): 236			

† turning 50 between 1950-2003 (Assumption: no death before age 50)
Source: WiDO (AOK)


number: ?
number: 64,000
Age—when—observation—begins T (t)
Lifespan X
number: 172,000
Age—at—observation—end T (t)+10
1950
2004
2013
time
birth period (length G)+50
observation period (length s)

In a simple random sample (i.i.d.) with measurements (V_i, W_i) it holds for the joint distribution

$$f(v_i, w_i; \theta) = f(v_i | w_i; \theta) f(w_i; \theta)$$

- Marginal likelihood inference : $\hat{\theta} := \arg\max_{\theta \in \Theta} \sum_{i=1}^{n} \log f(W_i; \theta)$
- In our example with $N := \{\mathbb{1}_{\{X \le x\}}, x \in \mathbb{N}_0\}$:
  1. Left truncation

$$(\underbrace{N(0), \ldots, N(T)}_{=:V}, \underbrace{N(T+1), \ldots, N(\infty)}_{=:W})$$

  2. Right censoring

$$(\underbrace{N(T+1), N(T+2)}_{=:W}, \underbrace{N(T+3), \ldots}_{=:V}, \underbrace{N(\infty)}_{=:W})$$

In a simple random sample (i.i.d.) with measurements (V_i, W_i) it holds for the joint distribution

$$f(v_i, w_i; \theta) = f(v_i | w_i; \theta) f(w_i; \theta)$$

- Conditional likelihood inference

$$\hat{\theta} := \arg\max_{\theta \in \Theta} \sum_{i=1}^{n} \log f(V_i | W_i; \theta)$$

- In our example:

$$(\underbrace{\mathbb{1}_{\{T+1 \le X\}}}_{=:W\ (\mathbb{1}_{\mathbb{T}})}, \underbrace{N(T+1), \ldots N(\infty)}_{=:V})$$

Assumptions:

- Let X be geometrically[1] distributed with true parameter value $\theta_0 \in [\xi, 1-\xi]$, i.e. X has the probability mass function:

$$f_G(\mathbf{x};\theta_0) = P_{\theta_0}\{X = \mathbf{x}\} = \begin{cases} \theta_0(1-\theta_0)^{\mathbf{x}-1}, & \text{if } \mathbf{x} \in \mathbb{N} \\ 0, & \text{if } \mathbf{x} \notin \mathbb{N} \end{cases}$$

- Enterprises can be observed from the year 2018 onwards

  i.e. at an age from $T+1$ onwards

  i.e. on the event $\mathbb{T} := \{T + 1 \leq X\}$

- No distributional assumption for T is needed.

[1] exponential, if time is continuous

The conditional (and marginal) Likelihood-contribution of an enterprise with a lifespan X, which can only be observed from the age $T+1$ onwards and conditional on the event $\mathbb{T} = \{T+1 \leq X\}$, and is observed up to an arbitrary, but fixed time $\chi \geq X$, is:

$$
\begin{aligned}
{}_TL(\theta) &= 1, \text{ if } \mathbb{1}_{\mathbb{T}} = 0 \\
{}_TL(\theta) &= \theta(1-\theta)^{(X-1)-T} \\
&= \prod_{x=T+1}^{\chi} \Big\{ \big(1 - P\{\text{obs. death in } x | \text{information at } x-1\}\big)^{1-\mathbb{1}_{\{\text{obs. death in } x\}}} \\
&\qquad \big(P\{\text{obs. death in } x | \text{information at } x-1\}\big)^{\mathbb{1}_{\{\text{obs. death in } x\}}} \Big\} \\
&= \prod_{x=T+1}^{\chi} \{(1 - \Delta\, {}_TA(x,\theta))^{1-\Delta\, {}_TN(x)} (\Delta\, {}_TA(x,\theta))^{\Delta\, {}_TN(x)}\}, \text{ if } \mathbb{1}_{\mathbb{T}} = 1
\end{aligned}
$$

- For enterprises that close after 2019, only survival up to and including 2019 is documented.
- Observation period covers $s := |\{2018, 2019\}| = 2$ years.
- marginal Likelihood-contribution conditional on $\mathbb{1}_{\mathbb{T}} = 1$:

$$
\begin{aligned}
{}_T L^c(\theta) &= \theta^{\mathbb{1}_{\{X \leq T+2\}}} (1-\theta)^{\min(X-1, T+2)-T} \\
&= \prod_{x=T+1}^{T+2} \Big\{ \big(1 - P\{\text{obs. death in } x | \text{information at } x-1\}\big)^{1-\mathbb{1}_{\{\text{obs. death in } x\}}} \\
&\qquad \big(P\{\text{obs. death in } x | \text{information at } x-1\}\big)^{\mathbb{1}_{\{\text{obs. death in } x\}}} \Big\} \\
&= \prod_{x=T+1}^{T+2} \{(1 - \Delta\, {}_T A^c(x,\theta))^{1-\Delta\, {}_T N^c(x)} (\Delta\, {}_T A^c(x,\theta))^{\Delta\, {}_T N^c(x)}\}
\end{aligned}
$$

- Fix t: $\Rightarrow \mathbb{1}_{\mathbb{T}}$ is still random - conditioning still necessary
- Marginal counting process ${}_tN := (0, \ldots, 0, N(t+1), \ldots N(\infty))$

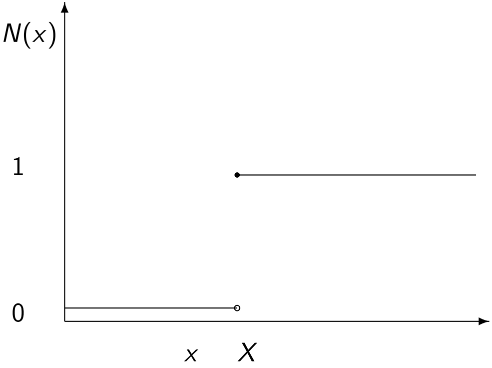
$\mathbb{1}_{\mathbb{T}} = 1$: $X \geq t + 1 \Rightarrow N(t) = 0$: ${}_tN$ is observable.

$\mathbb{1}_{\mathbb{T}} = 0$: $X \leq t \Rightarrow N(x) \equiv 1,\ x \geq t$: ${}_tN$ is observable!

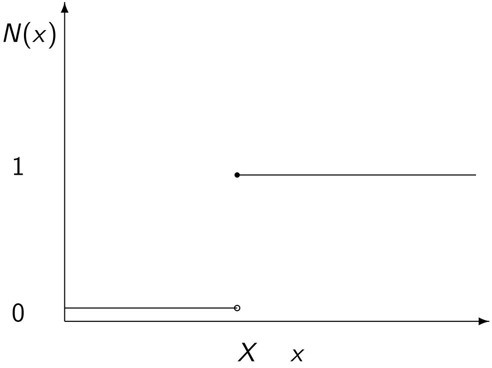
- Model 'time $\times$ life' in (joint) distribution of N: Increments

$N(x-1) \backslash N(x)$	0	1
0	0	1
1	not defined	0

- Want to integrate time-model in distribution-definition of N and ${}_tN$.
- (In vector 'usually' it is e.g. $E(N)$ componentwise.)
- Am I able to answer - at an age x - the question '$X = k$?' for every $k \leq x$?



- Want to integrate time-model in distribution-definition of N and ${}_tN$.
- (In vector ‘usually’ it is e.g. $E(N)$ componentwise.)
- Am I able to answer - at an age x - the question ‘$X = k$?’ for every $k \leq x$?



Concept memory and marginalization

- - i.e. to know at the age x whether the enterprise is closed yet, and if, when,
- - defines the filtration consisting of

$$\mathcal{F}_x := \sigma\{\mathbb{1}_{\{X=k\}}, 1 \leq k \leq x\}$$

- Note that $\mathcal{F}_x = \sigma\{\underbrace{\mathbb{1}_{\{X\leq k\}}}_{N(k)}, 1 \leq k \leq x\}$

Concept memory and marginalization

- ΔN with $\Delta N(x) := N(x) - N(x-1)$
- $E_\theta(\Delta N(x)|\mathcal{F}_{x-1}) = \theta \mathbb{1}_{\{X \geq x\}} =: \Delta A(x, \theta)$
- Later, when T is random, this will require X and T to be independent (which they never are).
- In the context of stochastic processes, the marginalization is reducing attention and corresponds to a coarser filtration.
- Later, increasing the filtration will be necessary when the random T introduces more information.
- Initiating attention from some age t onward (see also Figure), requires to exclude earlier outcomes from the probability model, formally represented by the set $\{\emptyset, 1 \leq k \leq t\}$.

Concept memory and marginalization

- To know at the age of $x \geq t+1$, whether the enterprise is closed yet ($\mathbb{1}_{\{X \geq t+1\}}$), and if, when (except is was before $t+1$)
- - is now

$$_t\mathcal{G}_x := \sigma\{ \underbrace{\underbrace{\mathbb{1}_{\{t+1 \leq X \leq k\}}}_{\mathbb{1}_{\{t+1 \leq X\}} N(k)}}_{=:\,_tN(k)}, \underbrace{\mathbb{1}_{\{X \geq k+1\}}}_{=:Y(k)}, t+1 \leq k \leq x\}.$$

- Note that the processes $\mathbb{1}_{\{X \geq k+1\}}$ can replace the information $\mathbb{1}_{\{X \geq t+1\}}$ because $\mathbb{1}_{\{X \geq k+1\}}$ for $k \geq t+1$ are in combination with $\mathbb{1}_{\{t+1 \leq X \leq k\}}$ redundant.
- In view of the Figure we can call this an *observed* filtration.

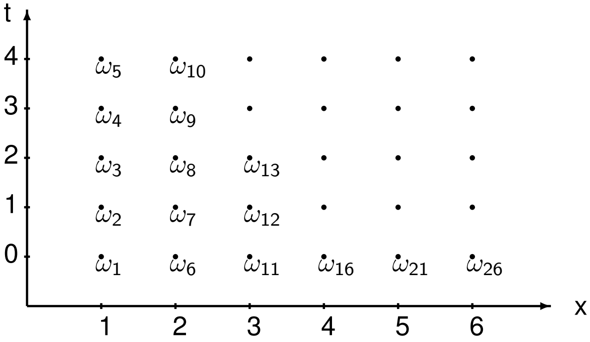
Concept memory and marginalization

$$\mathbb{1}_{\{t+1\leq \mathbf{x}\}}(1-\theta)^{\mathbf{x}-(t+1)}\theta + \mathbb{1}_{\{t\geq \mathbf{x}\}} = f^{\,{}_tN|\mathbb{1}_{\{t+1\leq X\}}}\left({}_tn|\mathbb{1}_{\{t+1\leq \mathbf{x}\}}\right)$$
$$= \prod_{x=t+1}^{\chi}(1-\Delta\,{}_ta(x,\theta))^{(1-\Delta\,{}_tn(x))}(\Delta\,{}_ta(x,\theta))^{\Delta\,{}_tn(x)}$$

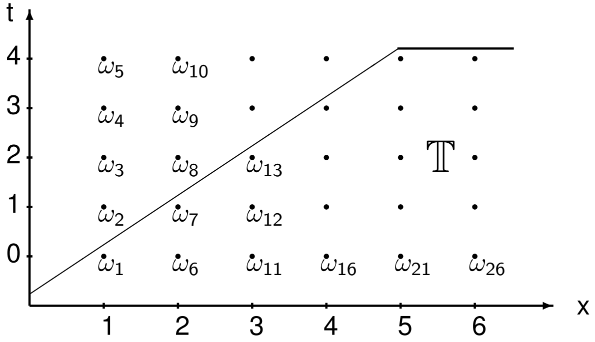
after ‘gauging’ ${}_tN(x) := N(x) - N(\min(x,t))$.

Let $\Omega := \{\omega_1, \omega_2, \ldots\}$ be the set of all possible events:

$$\omega_{(x-1)\cdot 5+t+1} := \{\text{enterprise has lifespan of } x \text{ years and has an age of } t \text{ in 2017}\},\ x \in \mathbb{N} \text{ and } t \in \{0, \ldots, 4\}$$



- Let $X : \Omega \to \mathbb{N} : \omega_{(x-1)\cdot 5+t+1} \mapsto x$ the lifespan of an enterprise and $T : \Omega \to \mathbb{N}_0 : \omega_{(x-1)\cdot 5+t+1} \mapsto t$ be the age in 2017
- Observability for an enterprise is only given on $\mathbb{T} := \{T + 1 \leq X\}$



- Ω can be combined with the σ-Algebra $\mathcal{F} = \mathcal{P}(\Omega)$ and the probability measure P_θ, where $\theta \in \Theta$, to the probability space $(\Omega, \mathcal{F}, P_\theta)$
- Assumption: year of foundation and lifespan are independent

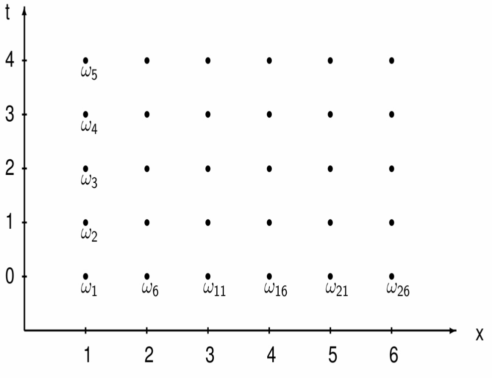
$$P_\theta(\omega) = P_X(\omega; \theta) P_T(\omega), \; \forall \omega \in \Omega, \; \theta \in \Theta$$

- semi-parametric model: $P_T(\omega)$ can be chosen arbitrarily
- Consideration of lifespan as a round game $\Rightarrow X$ is geometrically distributed with the true parameter value θ_0:

$$f_G(\mathbf{x}; \theta_0) := P_{\theta_0}\{X = \mathbf{x}\} = P_X\{X = \mathbf{x}; \theta_0\} = \begin{cases} \theta_0(1-\theta_0)^{\mathbf{x}-1}, & \text{if } \mathbf{x} = 1, 2, \ldots \\ 0, & \text{else} \end{cases}$$

- Let θ_0 be the probability for being closed within one year
- Assumption: θ_0 constant
- Aim: Derivation of a conditional (and marginal) Likelihood-function for the estimation of θ_0
- Therefore needed:
  - observable counting process ${}_T N$ and
  - its compensator ${}_T A$
  - with respect to an observable filtration $\{{}_T\mathcal{F}_x : x \in \mathbb{N}_0\}$

- Start with latent, not observable filtration, coarsening later
- Which questions about the enterprise should be able to be answered at an arbitrary, but fixed point of time $x \in \mathbb{N}_0$?
  (1) Does the enterprise have been closed until x?
  (2) When was the enterprise founded?
  (3) Is the enterprise observable? $\rightarrow$ measurability of $\mathbb{T}$ is essential


t
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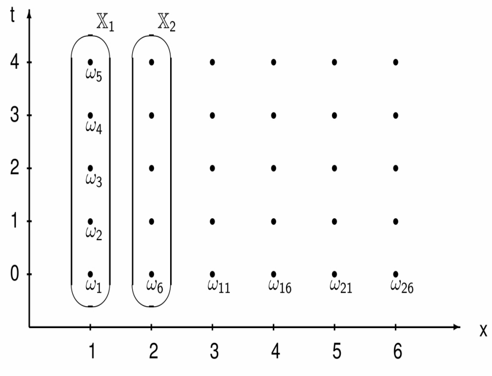
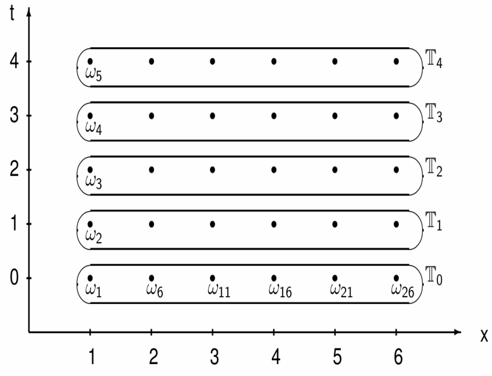


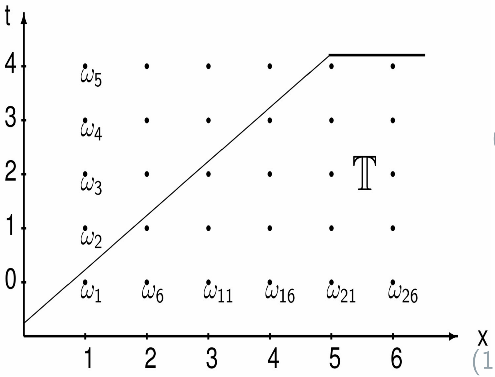
Q. (1): Information about closure in the first x years: $\mathcal{F}_x := \sigma\{\mathbb{X}_k : k \in \{1, \ldots, x\}\}$



Q. (1): Information about closure in the first x years: $\mathcal{F}_x := \sigma\{\mathbb{X}_k : k \in \{1, \ldots, x\}\}$

Q. (2): Information about year of foundation: $\mathcal{G}_0 := \sigma\{\mathbb{T}_t : t \in \{0, \ldots, 4\}\}$

(1), (2): $\mathcal{G}_x := \mathcal{F}_x \vee \mathcal{G}_0$



Q. (1): Information about closure in the first x years: $\mathcal{F}_x := \sigma\{\mathbb{X}_k : k \in \{1, \ldots, x\}\}$

Q. (2): Information about year of foundation: $\mathcal{G}_0 := \sigma\{\mathbb{T}_t : t \in \{0, \ldots, 4\}\}$

(1), (2): $\mathcal{G}_x := \mathcal{F}_x \vee \mathcal{G}_0$

Q. (3): Information about observability: $\mathcal{G}_T := \sigma\{\mathbb{T} \in \mathcal{F} : T \cap \{T = x\} \in \mathcal{G}_x, \forall x \in \mathbb{N}_0\}$ contains all information until T (because T is a $\mathcal{G}_x$-stopping time) It holds $\mathbb{T} \in \mathcal{G}_T$

(1) – (3): ${}_T\mathcal{G}_x := \mathcal{G}_x \vee \mathcal{G}_T$

- Independently of truncation, the value of ${}_{\mathbb{T}}N$ is observable, whereby

$$ {}_{\mathbb{T}}N(x) := \mathbb{1}_{\mathbb{T}} \mathbb{1}_{\{X \leq x\}} = \begin{cases} 1, & \text{if observable closure happened until time } x \\ 0, & \text{else} \end{cases} $$

- ${}_{\mathbb{T}}N$ is ${}_{\mathbb{T}}\mathcal{G}_x$-adapted and is called left-truncated counting process
- Analogously, ${}_{\mathbb{T}}Y$ is the ${}_{\mathbb{T}}\mathcal{G}_x$-adapted, left-truncated under-risk-process with:

$$ {}_{\mathbb{T}}Y(x) := \mathbb{1}_{\{T \leq x\}} \mathbb{1}_{\{X-1 \geq x\}} = \begin{cases} 1, & \text{if observable closure at } x+1 \text{ is possible} \\ 0, & \text{else} \end{cases} $$

- Aim: Decomposition of ${}_{\mathbb{T}}N = {}_{\mathbb{T}}A + {}_{\mathbb{T}}M$ into a ${}_{\mathbb{T}}\mathcal{G}_{x-1}$-adapted trend ${}_{\mathbb{T}}A$ (compensator) and a rest ${}_{\mathbb{T}}M$ (martingale with respect to $({}_{\mathbb{T}}\mathcal{G}_x, P_{\theta_0}^{\mathbb{T}})$)

Theorem (Andersen et al., 1988, Proposition 4.1)

If an enterprise is only observable on the event $\mathbb{T} = \{T + 1 \leq X\}$ *with* $\alpha_{\theta_0} := P_{\theta_0}\{\mathbb{T}\} > 0$, *then the related left truncated counting process* ${}_T N$ *has the compensator* ${}_T A$ *with* ${}_T A(0, \theta_0) = 0$ *and*

$$ {}_T A(x, \theta_0) = \theta_0 \sum_{k=1}^{x} {}_T Y(k - 1) $$

Here, ${}_T A$ *is defined with respect to a filtration* $\{{}_T \mathcal{G}_x : x \in \mathbb{N}_0\}$ *and conditional probability measure* $P_{\theta_0}^{\mathbb{T}}$ *with*

$$ P_{\theta_0}^{\mathbb{T}}\{\mathbb{G}\} := \frac{P_{\theta_0}\{\mathbb{G} \cap \mathbb{T}\}}{P_{\theta_0}\{\mathbb{T}\}} = \frac{P_{\theta_0}\{\mathbb{G} \cap \mathbb{T}\}}{\alpha_{\theta_0}}, \forall \mathbb{G} \in \mathcal{F}. $$

- We already have got the left-truncated counting process ${}_T N$ and its related $({}_T\mathcal{G}_x, P^{\mathbb{T}}_{\theta_0})$-compensator ${}_T A$
- Problem: $\{{}_T\mathcal{G}_x : x \in \mathbb{N}_0\}$ contains information, which is not observable in practical situations
- Example: A closure before the study begins is not observable, but for an $x \le T$ it holds, that:

$$\mathbb{X}_x \in {}_T\mathcal{G}_x$$

- Solution: coarser filtration $\{{}_T\mathcal{F}_x : T \le x\}$ with ${}_T\mathcal{F}_x \subseteq {}_T\mathcal{G}_x$ and:

$${}_T\mathcal{F}_x := \sigma\{{}_T N(k), {}_T Y(k) : k \in \{T, \dots, x\}\}$$

- ${}_T A$ also is the $({}_T\mathcal{F}_x, P^{\mathbb{T}}_{\theta_0})$-compensator of ${}_T N$

- Assumption: $\{X_1, \ldots, X_n\}$, $\{T_1, \ldots, T_n\}$ are sets of i.i.d. random variables
- marginal conditional likelihood-function for sample size n:

$$_T L_{\cdot}^{c}(\theta) = \prod_{i=1}^{n} \left[{_T L_i^c}(\theta) \right]^{\mathbb{1}_{\{T_i+1 \leq X_i\}}} = \prod_{i=1}^{n} \left[\theta^{\mathbb{1}_{\{X_i \leq T_i+2\}}} (1-\theta)^{\min(X_i-1, T_i+2)-T_i} \right]^{\mathbb{1}_{\{T_i+1 \leq X_i\}}}$$

- Point estimator for θ_0 (with $\chi \geq 6 \geq T_i + 2$):

$$\hat{\theta}_n = \frac{\sum_{i=1}^{n} \sum_{x=1}^{\chi} \Delta \; {_T N_i^c}(x)}{\sum_{i=1}^{n} \sum_{x=1}^{\chi} {_T Y_i^c}(x-1)}$$

Foundation year	Year of closure 2018 $x_j^{obs} - t_j^{obs} = 1$	 2019 $x_j^{obs} - t_j^{obs} = 2$	 2020 or later $m - m_{uncens}$
2013-2017	168	107	1,173
Total no. of observations (m): 1.4 mio			

No. observ. m	No. closures m_{uncens}	'Time at risk' (years) $\sum_{j=1}^{m_{uncens}}(x_j^{obs} - t_j^{obs}) + 2(m - m_{uncens})$	Point $\hat{\theta}$
1.4 mio	275	1·168 + 2·107+ 2·1,173	0.1009

Point estimator (practitioner's formula), $\hat{\theta}$:

$$\frac{\sum_{i=1}^{n} \mathbb{1}_{\{t_i \le \mathbf{x}_i - 1 \le t_i + 1\}}}{\sum_{i=1}^{n} \mathbb{1}_{\{t_i \le \mathbf{x}_i - 1\}} \left[(\mathbf{x}_i \wedge (t_i + 2)) - t_i\right]} = \frac{168{,}142 + 107{,}050}{1 \cdot 168{,}142 + 2 \cdot 107{,}050 + 2 \cdot 1{,}172{,}652} \approx 0.1009$$

Life expectancy: $\mathrm{E}_{\hat{\theta}}\{X\} = \frac{1}{\hat{\theta}} \approx 9.91$ years

No. observ. m	No. deaths m_{uncens}	'Time at risk' (years) $\sum_{j=1}^{m_{uncens}}(x_j^{obs} - t_j^{obs}) + 10(m - m_{uncens})$	Point $\hat{\theta}$
236	64	$111 + 10 \cdot 172$	0.035

Death hazard (theoretical formula), $\hat{\theta}$:

$$\frac{\sum_{i=1}^{n} \int_0^{53} d\ {}_T N_i^c(x)}{\sum_{i=1}^{n} \int_0^{54} {}_T Y_i^c(x) dx} = \frac{64}{111 + 10 \cdot 172} \approx 0,035$$

Life expectancy: $\mathrm{E}_{\hat{\theta}}\{X\} = \frac{1}{\hat{\theta}} \approx \frac{1}{0.035} \approx 28\ (+50 = 78)$ years

Theorem (Billingsley, 2012, Theo. 35.12)

Suppose that for $n \to \infty$

$$\sum_{x=1}^{G+s} E([\Delta\ {}_T N_\cdot^c(x) - \Delta\ {}_T A_\cdot^c(x)]^2 |\ {}_T\mathcal{F}_{x-1}^c) \xrightarrow{P} \sigma^2 > 0$$

and

$$\sum_{x=1}^{G+s} E[(\Delta\ {}_T N_\cdot^c(x) - \Delta\ {}_T A_\cdot^c(x)]^2 \mathbb{1}_{\{|\Delta\ {}_T N_\cdot^c(x) - \Delta\ {}_T A_\cdot^c(x)| \geq \epsilon\}}) \to 0$$

for each ϵ. *Then* $N_\cdot(G+s) - A_\cdot(G+s) \Rightarrow \sigma N$ *for* $n \to \infty$.

Corollary

Under Assumptions with $s \geq 2$ and $Z \sim N(0,1)$, it is $\sqrt{n}(\hat{\theta} - \theta_0) \Rightarrow \sigma^{-1} Z$, for $n \to \infty$.

Practitioner's Formula (Scholz and Weißbach (2024), F. 18):

$$\widehat{Var(\hat{\theta})} = \frac{\widehat{\sigma^{-2}}}{n} \approx \frac{\hat{\theta}(1-\hat{\theta})}{(\sum_{j=1}^{m_{uncens}} x_j^{\text{obs}} - t_j^{\text{obs}}) + s m_{cens}}$$

For time-continuous scale: Andersen et al. (1993): 'Statistical Models based on Counting Processes': Theorem VI.1.2

No. observ.	No. closures	'Time at risk' (years)	Point	SE
m	m_{uncens}	$\sum_{j=1}^{m_{uncens}}(x_j^{obs} - t_j^{obs}) + 2m_{cens}$	$\hat{\theta}$	$\sqrt{\widehat{Var}(\hat{\theta})}$
1.4 mio	275	1·168 + 2·107+ 2·1,173	0.1009	$1.8 \cdot 10^{-4}$

- Standard error:

$$\text{SE} \approx \sqrt{\frac{0.1009 \cdot (1 - 0.1009)}{1 \cdot 168{,}142 + 1 \cdot 107{,}050 + 2 \cdot 1{,}172{,}652}} \approx \sqrt{\frac{0.091}{2{,}727{,}546}} \approx 1.82 \cdot 10^{-4}$$

- Confidence interval$_{\alpha=5\%}$ for θ: $[0.1005; 0.1013]$
- ... for the German enterprise life expectancy: $[9.88; 9.95]$
- Width of interval: one months

No. observ.	No. deaths	'Time at risk' (years)	Point	SE
m	m_{uncens}	$\sum_{j=1}^{m_{uncens}}(x_j^{obs} - t_j^{obs}) + 10m_{cens}$	$\hat{\theta}$	$\sqrt{\widehat{Var}(\hat{\theta})}$
236	64	111+10·172	0.035	$1.4 \cdot 10^{-4}$

- Standard error (practitioner's formula: Weißbach et al. (2024, F. A7)):

$$\mathrm{SE} \approx \frac{\left(\sum_{i=1}^{n} \mathbb{1}_{\{t_i \leq \mathbf{x}_i < t_i+10\}}(\mathbf{x}_i - t_i) + 10m_{cens}\right)}{\sqrt{m_{uncens}}} = \frac{111{,}000 + 10 \cdot 172{,}000}{\sqrt{64{,}000}} \approx 1.4 \cdot 10^{-4}$$

- Confidence interval$_{\alpha=5\%}$ for θ: $[0.0350; 0.0355]$
- ... for the German people's life expectancy: $[28.10; 28.57] + 50 = [78.10; 78.57]$
- Width of interval: half year

- Andersen, P., Ø. Borgan, R. Gill und N. Keiding. 1993. *Statistical Models Based on Counting Processes*. Springer, New York.
- Billingsley, P. 2012. *Probability and Measure* (4th ed.). Wiley, New York.
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